

Logical normal forms for classes of bounded-degree finite graphs closed under substructure

Let K be a *well behaved* class of bounded-degree finite graphs – i.e. closed under substructure.

Definition: The *open diagram* of a finite structure A is the sentence which existentially quantifies one variable for each distinct element of A , describing the truth or falsity of each atomic formula. For example, the open diagram of the finite graph $A = \langle \{a_1, \dots, a_n\}, E^A \rangle$ is:

$$\begin{aligned} o_A \equiv & (\exists x_1 \dots x_n) \wedge \{x_i \neq x_j : 1 \leq i < j \leq n\} \wedge \\ & \wedge \{E(x_i, x_j) : A \models E(a_i, a_j), 1 \leq i < j \leq n\} \wedge \\ & \wedge \{\neg E(x_i, x_j) : A \not\models E(a_i, a_j), 1 \leq i < j \leq n\} \end{aligned}$$

Intuitively, o_A means “contains A ”, so $B \models o_A$ if and only if there is an embedding from A into B .

Note: K must also be closed under (finite) substructures of its pseudo-finite models because if a finite $A \subseteq B \models \text{Th}(K)$, then $B \models o_A$, so the open diagram of A is consistent with $\text{Th}(K)$. This means there must be a finite model $A' \models o_A$ with $A \subseteq A' \in K$. But K is closed under substructure, so $A \in K$.

Definition: The *closed diagram* of a finite structure A is the open diagram together with a clause stating there are no other elements connected to A . For example, the closed diagram of A is:

$$\begin{aligned} \chi_A \equiv & (\exists x_1 \dots x_n) \wedge \{x_i \neq x_j : 1 \leq i < j \leq n\} \wedge \\ & \wedge \{E(x_i, x_j) : A \models E(a_i, a_j), 1 \leq i < j \leq n\} \wedge \\ & \wedge \{\neg E(x_i, x_j) : A \not\models E(a_i, a_j), 1 \leq i < j \leq n\} \wedge \\ & [(\forall y) \wedge \{y \neq x_k : 1 \leq k \leq n\} \rightarrow \wedge \{\neg E(y, x_k) \wedge \neg E(x_k, y) : 1 \leq k \leq n\}] \end{aligned}$$

See that $B \models \chi_A$ if and only if there is a disjoint embedding from A into B — i.e. in such a way that no element of A is connected to anything outside of A .

Definition: We will say that set of first-order sentences N is *descriptive* for a class of finite structures K if for each A in K , there are sentences ε_A and δ_A in the Boolean closure of N such that $K \models \varepsilon_A \leftrightarrow o_A$ and $K \models \delta_A \leftrightarrow \chi_A$. Observe that these equivalences must necessarily also hold on all pseudo-finite models. I.e. for all models $B \models \text{Th}(K)$

1. $B \models \varepsilon_A$ if and only if A is embeddable in B , and
2. $B \models \delta_A$ if and only if A disjointly embeddable in B .

Definition: We say that a set of first-order sentences N is a *sentential normal form* over K if every first-order sentence ϕ is equivalent to a (finite) Boolean combination of sentences in N .

Theorem: N is a sentential normal form for K if and only if N is descriptive for K .

Proof: One direction is fairly trivial because, as we have illustrated above, there are first-order sentences that can express, for any finite model, whether it is embedded or disjointly embedded.

In the other direction, let $T = \text{Th}(K)$, and let the N -theory of A , $\text{Th}^N(A) = \{\psi \in N : A \models \psi\}$, be the sentences in N true of A . It suffices by the corollary on page 9 of [E&F] to show that any two models A and B of T with the same N -theory are elementarily equivalent.

If A and B are finite, then $A \models \varepsilon_A \Rightarrow B \models \varepsilon_A$ implies A is embeddable in B . Similarly, $B \models \varepsilon_B \Rightarrow A \models \varepsilon_B$ implies B is embeddable in A , hence A and B are isomorphic. (could use δ instead)

Suppose towards a contradiction that A is finite and B is infinite. Take any finite $A' \subseteq B$ larger than A . As observed earlier, K contains every finite substructure of its pseudo-finite models, so $A' \in K$. By hypothesis, $\varepsilon_{A'} \in \text{Th}^N(B)$ means $A \models \varepsilon_{A'}$, which is impossible because $|A'| > |A|$.

If A and B are infinite, it suffices to consider saturated models the size of the continuum (i.e. we assume the continuum hypothesis and choose to consider only ultraproducts over K , one for each complete theory extending T), aiming to show they are isomorphic. Notice that just as in the previous case, any finite substructure of A or B is a finite model in K . In particular, by satisfying the same closed diagrams (this is the only place δ is used), all finite components of A and B occur in the same quantity (finite or infinite). It remains to be seen that the infinite components also occur in the same quantity. The idea is to use limits of local neighborhoods to characterize each countable component. Suppose $k \times A' \subseteq A$, i.e. A contains k identical copies of the countably infinite component A' , and choose any $a \in A'$. For each radius r , let $N_r(a)$ be the r -neighborhood around a , which is finite because all our structures are bounded-degree. Let $A'' = k \times N_r(a)$ be the finite substructure of A , which as observed earlier must be in K . By hypothesis, $A \models \varepsilon_{A''} \Rightarrow B \models \varepsilon_{A''}$, which means that A'' is embedded in B . To see that the entire countable sub-model $k \times A'$ is embedded in B , we need to use a saturation argument. Let $\lambda^r(x)$ be a complete description of $N_r(a)$, and let $\mu^r(x_1 \dots x_k)$ be the formula that describes k identical disjoint such r -neighborhoods:

$$\mu^r(x_1 \dots x_k) \equiv \bigwedge \{d(x_i, x_j) > 2r : 1 \leq i < j \leq k\} \wedge \bigwedge \{\lambda^r(x_i) : i = 1, \dots, k\}$$

The k -type $\Lambda(x_1 \dots x_k) \equiv \{\mu^r(x_1 \dots x_k) : r = 0, 1, \dots\}$ is realized in B because it is finitely realized. Therefore, there are k disjoint copies of A' appearing in B . It is easy to see now that any countably graph is embedded the same number of times (finite or infinite) in A or B . To end the proof, it is necessary to observe that no type can be realized a countably infinite number of times (in an uncountably saturated model). To see this, let $\tau(x)$ be a type over a countably infinite subset A of $|A|$ and suppose $B = \{a : A \models \tau(a)\}$ were countably infinite. But then $\tau(x) \wedge \{x \neq b : b \in B\}$ would be a finitely consistent type over $A \cup B$ which is not consistent – contradiction.

Gaifman normal form (sentential) for bounded-degree graphs (inspired by pp. 31-32 of [E&F]).

Definition: Let G be the set of basic local sentences, and $\text{Th}^G(\mathfrak{A})$ the *Gaifman theory* of \mathfrak{A} .

Corollary: If C is a class of finite bounded-degree graphs, then any first-order sentence is equivalent over $T = \text{Th}(C)$ to a Boolean combination of basic local sentences.

Proof: By the theorem, it suffices to show that G can express the finite embedding conditions. For each finite A , the open diagram o_A is already a basic local sentence of radius zero. For the closed diagram, it is enough to express the disjoint embedding of k identical copies of each connected component C . Up to isomorphism, C can be described by a local formula $\gamma^{(d)}(x)$ of radius equal to the diameter d of C around some chosen center vertex x . So $k \times C$ is:

$$(\exists x_1 \dots x_k) \wedge \bigwedge \{d(x_i, x_j) > 2d : 1 \leq i < j \leq k\} \wedge \bigwedge \{\gamma^{(d)}(x_i) : i = 1, \dots, k\}$$